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77T-B135 JOHN L. GUSTAFSON, California Institute of Technology, Pasadena, California 91125. *A Fast-Convergence Infinite Product for Γ(z)*

An infinite product for $\Gamma(z)$ is developed from the Euler formula for $\Gamma(z)$; this product is combined with the classical product to give one with improved convergence. The composite product has an error term $O(N^{-2})$ compared to the $O(N^{-1})$ for the usual case, where N is the number of factors.

The Euler product is $\Gamma(z) = z^{-1}\,\Pi\,(1 + n^{-1})^{z}\,(1 + z\,n^{-1})^{-1}$ and the new product is
(*) $\Gamma(z) = z^{z-1}\,\Pi\,(1 + z\,n^{-1})^{-1}(1 - (z + n)^{-1})^{-z}$. Since the first-order error terms are equal and opposite they can be cancelled by geometric averaging:

$$\Gamma(z) = z^{z/2-1}\,\Pi\,[(1 + n^{-1})(1 - (z + n)^{-1})^{-1}]^{z/2}(1 + z\,n^{-1})^{-1}.$$

The following rapidly convergent product for π results by averaging Wallis' product with one derived by putting $z = ½$ in (*):

$$\pi^2 = 8\left(\frac{2\cdot 2\cdot 2\cdot 4}{1\cdot 3\cdot 3\cdot 3}\right)\left(\frac{4\cdot 4\cdot 4\cdot 6}{3\cdot 5\cdot 5\cdot 5}\right)\left(\frac{6\cdot 6\cdot 6\cdot 8}{5\cdot 7\cdot 7\cdot 7}\right)\cdots$$

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[*Full paper follows.*]

A FAST-CONVERGENCE INFINITE PRODUCT FOR THE Γ-FUNCTION

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Abstract

A new infinite product for the Γ-function is developed from the Euler formula for $\Gamma(z)$; this product is then combined with a well-known product to give a product with improved convergence properties. The composite product has an error term of $O(1/N^2)$, compared to $O(1/N)$ for the usual product formulas, where N is the number of terms.

Euler's formula is readily derivable from the definition of the Γ-function as follows:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt = \lim_{N\to\infty} \int_0^N \left(1 - \frac{t}{N}\right)^N t^{z-1} dt. \tag{1}$$

Integration by parts gives

$$\Gamma(z) = \lim_{N\to\infty} \frac{N! N^z}{z(z+1)(z+2)\cdots(z+n)} \tag{2}$$

Changing this to the form of an infinite product involves rewriting this in such a way as to eliminate the N^z factor. For example, Euler's infinite product may be obtained as follows:

$$\begin{aligned}
\Gamma(z) &= \frac{1}{z} \lim_{N\to\infty} \left(\frac{1}{z+1}\right)\left(\frac{2}{z+2}\right)\cdots\left(\frac{N}{z+N}\right) e^{z\log N} \\
&= \frac{1}{z} \lim_{N\to\infty} \left(\frac{1}{1+z}\right)\left(\frac{1}{1+z/2}\right)\cdots\left(\frac{1}{1+z/N}\right) \frac{e^{z(\log N - 1 - 1/2 - \ldots - 1/N)}}{e^{-z(1+1/2+\ldots+1/N)}} \\
&= \frac{1}{z} \lim_{N\to\infty} \left(\frac{e^{z}}{1+z}\right)\left(\frac{e^{z/2}}{1+z/2}\right)\cdots\left(\frac{e^{z/N}}{1+z/N}\right) e^{z(\log N - 1 - 1/2 - \ldots - 1/N)} \\
&= \frac{1}{z} e^{-\gamma z} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1+z/n}. \qquad (\gamma = .57721\cdots)
\end{aligned} \tag{3}$$

The N^{th} term of this product can be expanded as

$$\left(1 + \frac{z}{N} + \frac{z^2}{2N^2} + \ldots\right)\left(1 - \frac{z}{N} + \frac{z^2}{N^2} - \ldots\right) = 1 + \frac{z^2}{2N^2} + \ldots$$

Hence, the error term after N terms is $(1 + z^2/2N^2 + O(1/N^2))$. That is,

$$\Gamma(z) = \frac{1}{z} e^{-\gamma z} \left(\prod_{n=1}^{N} \frac{e^{z/n}}{1+z/n}\right)\left(1 + \frac{z^2}{2N} + O\left(\frac{1}{N^2}\right)\right). \tag{4}$$

The following method involves only regrouping to eliminate the N^z factor:

$$\Gamma(z) = \frac{1}{z}\lim_{N\to\infty}\left(\frac{1}{1}\right)^z\left(\frac{1}{1+z}\right)\left(\frac{2}{2}\right)^z\left(\frac{2}{2+z}\right)\cdots\left(\frac{N}{N}\right)^z\left(\frac{N}{N+z}\right)N^z$$
$$= \frac{1}{z}\lim_{N\to\infty}\left(\frac{2}{1}\right)^z\left(\frac{1}{1+z}\right)\left(\frac{3}{2}\right)^z\left(\frac{2}{2+z}\right)\cdots\left(\frac{N}{N-1}\right)^z\left(\frac{N}{N+z}\right)\cdot\left(\frac{N}{N}\right)^z.$$

Since $\lim_{N\to\infty}\left(\frac{N}{N+z}\right) = 1$, this becomes

(5) $$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^z\left(\frac{n}{n+z}\right) = \frac{1}{z}\prod_{n=1}^{\infty}\frac{(1+1/n)^z}{1+z/n}.$$

The N^{th} term of this product can be expanded as

$$\left(1+\frac{z}{N}+\frac{z(z-1)}{2N^2}+\ldots\right)\left(1-\frac{z}{N}+\frac{z^2}{N^2}-\ldots\right) = 1+\frac{z(z-1)}{2N^2}+\ldots$$

So

(6) $$\Gamma(z) = \left(\frac{1}{z}\prod_{n=1}^{N}\frac{(1+1/n)^z}{1+z/n}\right)\left(1+\frac{z(z-1)}{2N}+O\left(\frac{1}{N^2}\right)\right).$$

A similar approach produces a product that seems to be new to the literature:

$$\Gamma(z) = \frac{1}{z}\lim_{N\to\infty}\left(\frac{1}{1+z}\right)\left(\frac{2}{2+z}\right)\cdots\left(\frac{N}{N+z}\right)N^z$$
$$= \frac{1}{z}\lim_{N\to\infty}\left(\frac{1/z}{\left(1+\frac{1}{z}\right)^{1-z}\left(1+\frac{1}{z}\right)^z}\right)\left(\frac{2/z}{\left(1+\frac{2}{z}\right)^{1-z}\left(1+\frac{2}{z}\right)^z}\right)\cdots\left(\frac{N/z}{\left(1+\frac{N}{z}\right)^{1-z}\left(1+\frac{N}{z}\right)^z}\right)N^z$$
$$= \frac{1}{z}\lim_{N\to\infty}\left(\frac{1/z}{\left(1+\frac{0}{z}\right)^{1-z}\left(1+\frac{1}{z}\right)^z}\right)\left(\frac{2/z}{\left(1+\frac{1}{z}\right)^{1-z}\left(1+\frac{2}{z}\right)^z}\right)\cdots\left(\frac{N/z}{\left(1+\frac{N-1}{z}\right)^{1-z}\left(1+\frac{1}{z}\right)^z}\right)\left(\frac{N^z}{\left(1+\frac{N}{z}\right)^z}\right).$$

.

Now, $\lim_{N\to\infty}\left(\frac{N}{1+N/z}\right)^z = z^z$. So

(7) $$\Gamma(z) = z^{z-1}\prod_{n=1}^{\infty}\left(\frac{1}{1+z/n}\right)\left(\frac{1}{1-1/(z+n)}\right)^{z}.$$

The N^{th} term of this product can be expanded as

$$\left(1-\frac{z}{N}+\frac{z^2}{N^2}-\ldots\right)\left(1+\frac{z}{N+z}+\frac{z(z-1)}{2(N+z)^2}+\ldots\right)=1-\frac{z(z-1)}{2N^2}+\ldots$$

Therefore, the error term is $1-\frac{z(z-1)}{2N}+O\left(\frac{1}{N^2}\right)$.

The error terms for (5) and (7) have $O(1/N)$ terms that are opposite in sign, and hence may be used to cancel one another by averaging, leaving only $O(1/N^2)$ terms in the error. The deep reason for this apparent coincidence is not clear, and it indeed may only be a coincidence. This is a subject to be pursued, as is the question of whether this technique may be continued to eliminate higher-order terms. It also seems that this method would work on infinite products in general, not just the Γ-function.

Before averaging the two products, it is necessary to note where they are valid. Equations (1) through (7) are valid for all z in the complex plane, except at the poles of $\Gamma(z)$, $z = 0, -1, -2, \cdots$. Hence, we may average term by term. Geometric averaging is more effective than arithmetic averaging in this case, reducing the size of the remaining error terms. Multiplying the two products term by term and taking the square root, we obtain

(8) $$\Gamma(z) = z^{z/2-1}\prod_{n=1}^{\infty}(1+1/n)\left(\frac{1}{1-1/(z+n)}\right)^{z/2}\left(\frac{1}{1+z/n}\right)$$

The error term after N terms, to $O(1/N^4)$, is

$$1+\frac{z(z-1)(z-2)}{2N^2}\left(\frac{1}{6}-\frac{z}{6N}+O\left(\frac{1}{N^2}\right)\right).$$

If formula (8) is truncated after a finite number of terms, it may be seen that the formula is exact for $z = 1$ and $z = 2$, and is an excellent approximation for z in the neighborhood of 0, 1, and 2. It achieves an accuracy with 100 terms that the standard infinite products (3) and (5) require about 50000 terms to achieve.

Several interesting formulas may now be derived as special cases of (8). For example, an improved version of Wallis' product may be found; from (7):

$$(9)\qquad \left(\Gamma\left(\tfrac{1}{2}\right)\right)^2 = \pi = \tfrac{1}{2}^{2\left(\frac{1}{2}-1\right)} \prod_{n=1}^{\infty} \left(\frac{1}{1+\frac{1}{2}/n}\right)^2 \left(\frac{1}{1-1/\left(\frac{1}{2}+n\right)}\right) = 2\prod_{n=1}^{\infty} \left(\frac{(2n)^2}{(2n-1)(2n+1)}\right)$$

$$= 2\left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\ldots$$

This is the usual form of the Wallis product. Applying (5),

$$(10)\qquad \left(\Gamma\left(\tfrac{1}{2}\right)\right)^2 = \pi = \tfrac{1}{2}^{-2} \prod_{n=1}^{\infty} \left(\frac{1}{1+\frac{1}{2}/n}\right)^2 (1+1/n) = 4\prod_{n=1}^{\infty} \left(\frac{(2n)(2n+2)}{(2n+1)^2}\right)$$

$$= 4\left(\frac{2\cdot 4}{3\cdot 3}\right)\left(\frac{4\cdot 6}{5\cdot 5}\right)\left(\frac{6\cdot 8}{7\cdot 7}\right)\ldots$$

Equations (9) and (10) are obviously identical except for the grouping of the factors; this occurs whenever z is rational. However, note that (9) approaches π from below whereas (10) approaches π from above. Multiplying the two,

$$(11)\qquad \pi^2 = 8\left(\frac{2\cdot 2\cdot 2\cdot 4}{1\cdot 3\cdot 3\cdot 3}\right)\left(\frac{4\cdot 4\cdot 4\cdot 6}{3\cdot 5\cdot 5\cdot 5}\right)\left(\frac{6\cdot 6\cdot 6\cdot 8}{5\cdot 7\cdot 7\cdot 7}\right)\ldots$$

This converges remarkably fast, giving π = 3.14 with only five terms. The usual Wallis product requires 120 terms to get these three decimal places of accuracy.

Similar expressions for the Γ-function of other rational numbers are easily found (rationality simply permits simplification to an expression involving integers); for example,

$$(12)\qquad \left(\Gamma\left(\tfrac{1}{3}\right)\right)^6 = 243\left(\frac{3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 6}{1\cdot 4\cdot 4\cdot 4\cdot 4\cdot 4}\right)\left(\frac{6\cdot 6\cdot 6\cdot 6\cdot 6\cdot 9}{4\cdot 7\cdot 7\cdot 7\cdot 7\cdot 7}\right)\ldots$$

$$(13)\qquad \left(\Gamma\left(\tfrac{3}{4}\right)\right)^8 = \frac{1024}{243}\left(\frac{4^5\cdot 8^3}{3^3\cdot 7^5}\right)\left(\frac{8^5\cdot 12^3}{7^3\cdot 11^5}\right)\left(\frac{12^5\cdot 16^3}{11^3\cdot 15^5}\right)\ldots$$

and in general, using equation (8),

$$(14)\qquad \left(\Gamma\left(\frac{a}{b}\right)\right)^{2b} = \left(\frac{b}{a}\right)^{2b-a} \prod_{n=1}^{\infty} \frac{(nb)^{2b-a}(nb+b)^a}{(nb-b+a)^a (nb+a)^{2b-a}}$$

Infinite products for the Γ-function for imaginary argument are most conveniently done by separating modulus and argument. The improved convergence formulas are

$$(15)\qquad \left|\Gamma(\alpha i)\right|^2 = \frac{e^{-\pi\alpha/2}}{2} \prod_{n=1}^{\infty} \frac{(\alpha n)^2}{(\alpha n)^2+1} e^{\arctan\left(\frac{\alpha}{n^2-n+\alpha}\right)}$$

and

$$\arg\left(\Gamma\left(\alpha i\right)\right)=\frac{\alpha\log\alpha-\pi}{2}+\sum_{n=1}^{\infty}\left[\frac{\alpha}{2}\left[\log\left(\frac{1+1/n}{\left(n-1\right)^2+\alpha^2}\right)+\tfrac{1}{2}\log\left(\left(n^2+\alpha^2-n\right)^2+\alpha^2\right)\right]-\arctan\left(\frac{\alpha}{n}\right)\right] \tag{16}$$

for α real and positive.

Conclusions

The product formula for $\Gamma(z)$ can be greatly improved in convergence properties by a technique that amounts to little more than rearrangement followed by multiplication by itself. As an algorithm, the resulting infinite product has several distinctive features:

(1) It is exact for all z except the poles of $\Gamma(z)$; that is, it is not an asymptotic formula like the Stirling formula.

(2) In numerical computation it requires no storage or computation of coefficients, as do polynomial approximations, continued fraction methods, the Stirling series, and so on.

(3) It is fast enough to be used for numerical computation of $\Gamma(z)$, and in fact is well suited to small calculators with limited storage.

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